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even numbered subscript is identically zero, and in order that the remaining C's may be zero, the equation of the problem must hold.

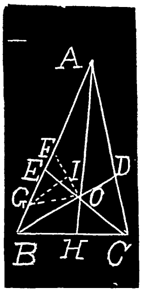
GEOMETRY.

123. Proposed by P. C. CULLEN, Indianola, Iowa.

If the bisectors of two angles of a triangle are equal, those angles are equal, and the triangle is isosceles.

Another Demonstration by JOHN G. GREGG, Terre Haute, Indiana.

Let ABC be the given triangle, and BD, CE, and AH the three bisectors of its angles meeting in O, and let $BD=CE$. We are to show that $\angle ABC=\angle ACB$. If these angles are not equal suppose $\angle ACB>\angle ABC$, then will AB be greater than AC. Take $AG=AC$, and $AF=AD$; then will $OG=OC$ and $OF=OD$, and $\angle GOF=\angle COD=\angle BOE$. Also $OB+OF=BD$...(1), and $OG+OE=CE$...(2).



It can be established that E will always fall between G and F. OB is greater than OG, and OF is greater than, equal to, or less than OG. If $OF>OG$, then also $OF>OE$, and $OB+OF>OG+OE$, or by (1) and (2), $BD>CE$. But by hypothesis, $BD=CE$. Hence $\angle ACB=\angle ABC$. Q. E. D.

Again, if OF is equal to or less than OG, draw GI making $\angle OGI=\angle OBA$. Obviously I will fall between O and F, and the triangles OBE and OGI are similar. Then since $BO>OG$, we have $BO-OE>OG-OI$, and much more, $BO-OE>OG-OF$. Hence $BO+OF>OG+OE$, and from (1) and (2), $BD>CE$ as before, and the theorem is established.

Corollary 1. If two lines BD and CE are drawn through a point O in the bisector of an angle, and meeting the sides of the angle, the one (BD) making the less angle with the bisector is the greater.

Corollary 2. If the two lines BD and CE make equal angles with the bisector, they are equal.

Corollary 3. The line through O, perpendicular to the bisector, is a minimum.

Corollary 4. Two triangles are equal if their bases, the angles opposite the bases, and the bisectors of those angles are respectively equal.

320. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Prove by plane geometry the following interesting theorem:

If from a point in the plane of a triangle perpendiculars are demitted upon the three sides of the triangle, and if the area of the triangle formed by connecting the feet of these perpendiculars is denoted by $\triangle'$, *the distance of the assumed point from the center of the circle circumscribed about the original triangle by* R', *the radius of the circumscribed circle by* R, *and the area of the pedal triangle by* $\triangle$, *then will* $\triangle'/\triangle=\pm[(R^2-R'^2)/R^2]$.

I. Solution by the PROPOSER.

There is an error in the statement of this theorem, which, I am afraid, was made by me through an oversight. Instead of the pedal triangle $\triangle$ it should be the triangle whose vertices are the mid-points of the sides of the original triangle. I saw this theorem for the first time, stated but not proved, in a German collection of geometrical exercises, with the following note: "In the English periodical, the *Mathematician*, anno 1844, the editor says that it would be desirable that this remarkable theorem should be established by pure geometry, which would form a good exercise for the student; and that he would be glad to give its investigation, in this way, in some future number of the *Mathematician*, should such be furnished by any of his correspondents." The *Mathematician*, however, has never furnished such a purely geometrical proof. For years I have tried to discover one, but I never succeeded in finding any other proof than the following trigonometrical one.

Let ABC represent the triangle, O the center of its circumcircle, P the assumed point; PL, PM, PN the perpendiculars let fall from it upon the sides of the triangle; OD perpendicular to AC, and OE to AB; PF perpendicular to OD, and PG to OE. Denote the angles of the triangle ABC by A, B, C; $\angle ACP$ by θ, $\angle ABP$ by ϕ, $\angle OPF$ by α, $\angle OPG$ by β.

$\dfrac{MN}{CM}=\dfrac{\sin C}{\cos\theta}$, $\dfrac{LN}{BL}=\dfrac{\sin B}{\cos\phi}$, but $CM=R\sin B+R'\cos\alpha$, and $BL=R\sin C+R'\cos\beta$.

$\therefore MN=\dfrac{\sin C}{\cos\theta}(R\sin B+R'\cos\alpha)$. Let $N=\dfrac{\sin B}{\cos\phi}(R\sin C+R'\cos\beta)$.

$\therefore \triangle'=\tfrac{1}{2}MN\times LN\sin(\theta+\phi)=\tfrac{1}{2}.\dfrac{\sin B\sin C}{\cos\theta\cos\phi}\sin(\theta+\phi)\ (R\sin B+R'\cos\alpha)\times(R\sin C+R'\cos\beta)=\tfrac{1}{2}\sin B\sin C(\tan\theta+\tan\phi)\ (R\sin B+R'\cos\alpha)\ (R\sin C+R'\cos\beta)\ldots(1)$.

But in $\triangle OPC$ and $\triangle OPB$, $R':R=\cos(B+\theta):\sin(\theta+\alpha)\ldots(2)$.
$R':R=\cos(C+\phi):\sin(\phi+\beta)\ldots(3)$, and $\alpha+\beta=A\ldots(4)$.

From (2), $\tan\theta=\dfrac{R\cos B-R'\sin\alpha}{R\sin B+R'\cos\alpha}\ldots(5)$, and from (3), $\tan\phi=\dfrac{R\sin C-R'\sin\beta}{R\sin C+R'\cos\beta}\ldots(6)$.

Substituting these values in (1), we obtain

$$\triangle'=\tfrac{1}{2}\sin B\sin C[R^2\sin A+RR'\cos(B+\beta)+RR'\cos(C+\alpha)-R'^2\sin A].$$

But since $\alpha+\beta=A$, $B+\beta=B+A-\alpha=180°-(C+\alpha)$.

$\triangle'=\tfrac{1}{2}\sin A\sin B\sin C(R^2-R'^2)$. Since $\sin A=8\triangle/bc$, $\sin B=8\triangle/ac$, $\sin C=8\triangle/ab$, $\sin A\sin B\sin C=512\triangle^3/a^2b^2c^2=2\triangle/R^2$.

$\therefore \triangle'=\dfrac{\triangle}{R^2}(R^2-R'^2)$, or $\dfrac{\triangle'}{\triangle}=\dfrac{R^2-R'^2}{R^2}$. Q. E. D.

Dr. Zerr gave a general discussion of the problem as proposed and also of the problem as intended by the proposer. We give below Professor Schmall's demonstration of the problem as proposed.

II. Solution by C. N. SCHMALL, A. B., 89 Columbia Street, New York City.

In Fig. 1, let XYZ be the pedal triangle. Then $AYGZ$ being a cyclic quadrilateral, AG is clearly the diameter of the circle, and we have,

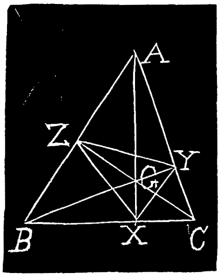


Fig. 1.

$$YZ=AG\sin A=AY\csc AGY\sin A=c\cos A\csc C\sin A$$
$$=a\cos A=R\sin 2A\ldots(1).$$

Similarly, $ZX=b\cos B=R\sin 2B$, and

$$XY=c\cos C=R\sin 2C.$$

$\therefore$ perimeter $XYZ=\Sigma(a\cos A)=R\,\Sigma\sin 2A$.

Also $\angle YXZ=\angle ABY+\angle ACZ=180^\circ-2A$.

$\therefore 2$ area $XYZ=XY.XZ.\sin YXZ=bc.\cos B\cos C\sin 2A$.

$\therefore$ area $XYZ=\frac{1}{2}bc\cos B\cos C.2\sin A\cos A=2\triangle''\Pi\cos A$

[where $\triangle''$=area of triangle ABC]$=\dfrac{\Pi(a\cos A)}{2R}=\dfrac{\Pi(R\sin 2A)}{2R}$ [by (1) above]

$=\dfrac{R^3\Pi(\sin 2A)}{2R}=\frac{1}{2}R^2\Pi\sin 2A=\triangle$ =area of the *pedal triangle* of triangle ABC.

Now, referring to Fig. 2, let P be the given point (taken within the triangle ABC for convenience).

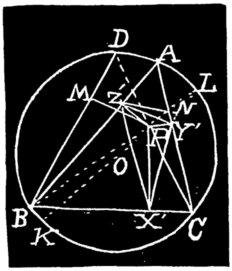


Fig. 2.

Let ABC be the given triangle; O the center of its circumscribed circle; X', Y' and Z' the feet of the perpendiculars from P on the sides of the triangle. Let CP (produced) meet the circle in D. Draw BD, and draw PM perpendicular to BD; draw also $Z'N$ perpendicular to $X'Y'$.

Now since $PX'CY'$ is cyclic, and PC is the diameter of its circle, we have

$$X'Y'=PC\sin C\ldots(1),$$

(see Phillips and Strong's *Trigonometry*, p. 50; ex. 10); similarly in the quadrilateral $BZ'PX'$,

$$Z'X'=PB\sin B\ldots(2).$$

Also, from the similar right triangles $Z'X'N$ and PBM [which are similar because $\angle Z'X'N=\angle Z'X'P+\angle Y'X'P=\angle Z'BP+\angle Y'CP=\angle Z'BP+\angle DBA=\angle MBP$] we have

$$\frac{Z'N}{PM}=\frac{Z'X'}{PB}=\sin B\ [\text{by } (2)]\ldots(3).\quad \text{Also, } \frac{PM}{PD}=\sin D=\sin A\ldots(4).$$

Multiplying equations (1), (3), and (4) together, we get,

$Z'N.X'Y'=PC.PD.\sin A\sin B\sin C$, *i. e.*, 2 area $X'Y'Z'=PC.PD.\Pi\sin A$

$=PL.PK.\Pi\sin A=(OL-OP)(OL+OP).\Pi\sin A$

$=(OL^2-OP^2).\Pi\sin A=(R^2-R'^2).\Pi\sin A.$

If the point P be taken outside the circle of the triangle ABC, then $(OL-OP)$ is negative and thus we get the double sign.

$\therefore \triangle'=\frac{1}{2}(R^2\pm R'^2).\Pi\sin A$. Hence,

$$\frac{\triangle'}{\triangle}=\frac{\pm\frac{1}{2}(R^2-R'^2).\Pi\sin A}{\frac{1}{2}R^2.\Pi\sin 2A}=\pm\frac{(R^2-R'^2}{R^2}\cdot\frac{\Pi\sin A}{\Pi\sin 2A}.$$

Thus the result, as given by the proposer, is not quite right.

He probably misused the well known formula, $\Sigma\sin 2A=4\,\Pi\sin A$.

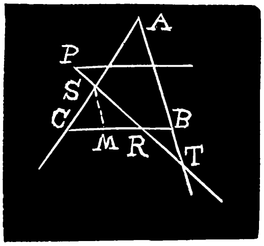
COROLLARY. If the point P lie on the circumscribed circle of triangle ABC, the points X', Y', Z' lie on "Simson's Line" and $\triangle'=0$.

321. Proposed by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Texas.

ABC is an isosceles triangle. Through any point P in its plane draw a line $PSRT$ cutting the sides AC, CB, AB in the points S, R, and T, respectively (R between B and C), so that the segments CS and BT shall be equal.

Solution by C. N. SCHMALL, 89 Columbia Street, New York City.

In the figure, let SM be drawn parallel to AB. Then, if $CS=BT$, $SR=RT$.



Hence to locate the point R we must solve the problem: Given a line drawn through a fixed point, and cutting two fixed intersecting lines, to find the locus of the middle point of the intercepted segment.

Here, we then have $PT-PS=2(PR-PS)$, or $PT+PS=2PR$.

Let (ρ', θ') be the polar co-ordinates of P, and let the equations of AC and AB be

$$a_1x+b_1y+c_1=0\ldots(1),\quad a_2x+b_2y+c_2=0\ldots(2).$$

Transforming to polar co-ordinates these become

$$a_1\rho\cos\theta+b_1\rho\sin\theta+c_1=0\ldots(3),\ \text{and}\ a_2\rho\cos\theta+b_2\rho\sin\theta+c_2=0\ldots(4).$$

From (3), $\rho=-\dfrac{c_1}{a_1\cos\theta+b_1\sin\theta}=PS$; from (4), $\rho=-\dfrac{c_2}{a_2\cos\theta+b_2\sin\theta}=PT$.

Hence the polar equation for the locus of P is, $2\rho=2PR=PT+PS$.

$$\therefore 2\rho=-\frac{c_1}{a_1\cos\theta+b_1\sin\theta}-\frac{c_2}{a_2\cos\theta+b_2\sin\theta}$$